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Recursive Numerical Algorithm for Conformal Mapping in Two-Dimensional Hydrodynamics

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An original recursive numerical method (RT algorithm) for conformal mapping is presented. Based on the Lavrentiev variational principle, it is a powerful means for the solution of various two-dimensional boundary value problems. The RT algorithm can be used to solve internal and external hydrodynamics problems occurring in spacecraft research, namely the calculation of fluid motions with local vorticity regions. For external problems, conformal mapping of the fluid flow area onto the outside of the unit circle is constructed; for internal problems the area is mapped onto one of standard domains (the unit circle or circular annulus). Fluid flows in a cylindrical tank with inner radial damping ribs and nearby an air brake with sharp edges with or without free eddies are discussed as examples.

I. Introduction

CONFORMAL mapping is widely used to generate orthogonal grids. At the same time it provides a powerful approach to two-dimensional mathematical physics problems with complicated boundary conditions. The combination of conformal mapping with a variational technique provides a universal tool for high-precision solution of various types of boundary value problems. The practical application of this approach has been limited by the fact that exact conformal maps are known only for certain simple domains, whereas for most common forms they must be computed numerically. The first practical conformal mapping procedure was presented by Theodorsen[1] and Theodorsen and Garrick.[2] That method and its numerous generalizations were based on mapping a standard domain (such as the unit circle) onto the initial domain. Methods of this type are based on an integral equation, which is solved by a fixed-point iteration. Another class of numerical conformal mapping methods deals with mapping from the given region to a standard domain. Methods of this type are based on the Symm equation, a linear Fredholm integral equation of the first kind.[3] Typically the integral equation reduces to an algebraic system.

Two special problems are important in mapping of a given domain onto a standard domain. First is the local issue of the treatment of singularities in the mapping function for regions with corners. For such domains the use of preliminary analytical transformations is needed. The second problem, the global issue in numerical conformal mapping, is the phenomenon known as crowding. This effect is clearly seen when an elongated region (such as a rectangle with a large aspect ratio) is mapped directly onto the unit circle. Another technique, the mapping of polygons by the Schwarz–Christoffel formula and its relatives, is effective for certain simple polygonal regions.[4] The iterative method for conformal mapping of a simply connected domain (unit circle for example) onto a region with smooth boundary and of a circular annulus onto a doubly connected domain with smooth boundary curves was given by Wegmann.[5,6] The preceding topics were all reviewed in Ref. 7.

During the last decade, numerical conformal mapping has been developed intensively. Parallel computers (e.g., Cray) whose architectures support powerful operations can provide certain numerical conformal mapping methods related to huge linear equation systems. These methods mainly use rapid elliptic solvers to solve those systems.[8] In hydrodynamics conformal mappings are widely applied to study the flow near airfoils with sharp trailing edges.[9,10] Another area of application is computer generated computational grids. In some investigations the numerical scheme is based on the conjunction of the boundary integral technique with the covariant Laplace equation method for conformal mapping.[11] One more approach to grid generation was suggested by Moretti.[12] His modification of the classical Theodorsen technique for conformal mapping was based on using an auxiliary contour, closer to the circle than the given contour. A preliminary Kármán–Trefftz transform was used for mapping contours with corners.

The original method of numerical conformal mapping (RT algorithm) presented herein was devised by Rabinovich and Tyurin.[13,14] The RT algorithm is a recursive numerical algorithm for conformal mapping of a simply connected domain onto the unit circle and of a doubly connected domain to the circular annulus. It is based on the Lavrentiev variational principle and includes multiple calculations of Fourier expansion coefficients. No algebraic system solution is used in the RT algorithm. Unlike traditional Theodorsen-type and Symm-type methods, in the present method both mappings—direct (from the given region onto a standard domain) and inverse (from a standard domain onto the given area)—are built simultaneously. At each iteration step of the external T-procedure a computational error of the inverse mapping is reduced to zero by means of the internal R-procedure. The mapping at each step is represented by Taylor (for simply connected domains) or Laurent (for doubly connected domains) series. The final result is expressed as an array of coefficient sets. Both problems mentioned earlier (corner singularities and crowding) can be solved by using this algorithm.

The RT algorithm has been thoroughly tested on many regions with known analytical or numerical mappings. As examples of simply connected domains with smooth contours, ellipses with various axis ratio and epithrohoids of various powers have been chosen; a cross and a circle with sectorial cutouts have been taken as examples of simply connected domains with corners. Various types of airfoils are also used as test cases. Eccentric elliptical annuli with

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various aspect ratios have been selected to verify RT mapping of doubly connected regions with smooth contours. Circular annuli with cutouts and outer areas of cross sections of electromagnet and ferromagnetic rail with an air gap are the typical examples of the doubly connected domains with corners.[15]

This list of transformed domains demonstrates clearly the main aim of the present investigation, the preparation of the given areas for later application of variational or series expansion methods. Therefore, convergence by L_2-norm for contours is used, instead of collocation type convergence. The RT algorithm has proved its effectiveness for various hydrodynamics and elasticity theory problems[16,17] and has been implemented in a multipurpose computer software pack (RT-Soft). It provides the means for constructing orthogonal grids for complicated objects and can be combined easily with conventional methods used to solve boundary value problems, e.g., with Ritz, Galerkin, and finite difference methods.

Spacecraft research is one of the fields that requires the most efficient and accurate solution of the adjacent boundary value problems.[18,19] Some of the corresponding questions are discussed next, in particular, internal and external hydrodynamics problems of fluid motions with local vorticity regions.

II. Lavrentiev Variational Principle of Conformal Mapping

Let $z = re^{i\varphi}$ be the complex coordinate of a plane containing a closed line C and $w = \rho e^{i\theta}$ be the complex coordinate of a plane containing a unit circle Γ. Their origin is located in the center of the circle, and the real axes of the two planes are made to coincide.

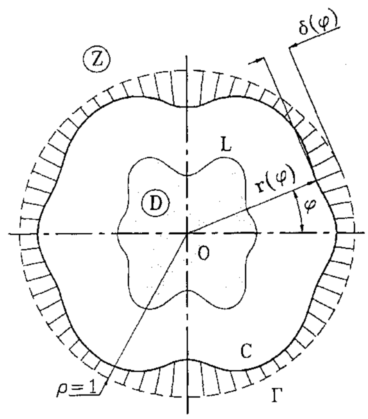


Fig. 1 Boundary variation of the initial domain demonstrating the Lavrentiev variational principle.

Assume that the contour C is very close to the circle G (Fig. 1):

$$|\delta(\varphi)| < \varepsilon; \qquad |\delta'(\varphi)| < \varepsilon$$
$$|\delta''(\varphi)| < \varepsilon; \qquad (0 \leq \varphi \leq 2\pi) \tag{1}$$

where $\delta(\varphi) = 1 - r(\varphi)$, r and φ are the radius vector and the polar angle of a point on C, and ε is the small positive number $\varepsilon \ll 1$.

The Fourier transform of the function $\delta(\varphi)$ may be presented as

$$\delta(\varphi) = \varepsilon\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos n\varphi + b_n \sin n\varphi)\right] \tag{2}$$

In accordance with the Lavrentiev principle,[20,21] the conformal mapping of the contour C onto the unit circle Γ and the inverse mapping of Γ onto C are defined by the functions

$$w = z\left[1 + \varepsilon\left(a_0 + \sum_{n=1}^{\infty} c_n z^n\right)\right]$$
$$z = w\left[1 - \varepsilon\left(a_0 + \sum_{n=1}^{\infty} c_n w^n\right)\right]; \qquad c_n = a_n - ib_n \tag{3}$$

These expressions are convenient for numerical implementation.

The transformation $z = Z(w)$, given by the second formula (3), is in fact the same as the first step of the Theodorsen mapping. It is clear from the following relations:

$$\ln r = \ln[1 - \delta(\varphi)] = -\delta(\varphi) + \mathcal{O}(\varepsilon^2), \qquad \varphi - \theta = \mathcal{O}(\varepsilon^2) \tag{4}$$

These relations are the direct result of the conditions (1).

III. Mapping of a Quasicircular Domain onto a Circle (R-Procedure)

On the basis of the Lavrentiev principle we can set up the algorithm for mapping a quasicircular domain onto a circle, as well as the inverse mapping, with any given accuracy. Let the contour C be close to the circle Γ according to inequalities (1) with the tolerance $\mathcal{O}(\varepsilon)$. A number of points N equal to some power of 2 are chosen on the circle Γ. As a first guess the given values of $r(\varphi)$ are interpolated at each value φ. Equation (2) is used to find the coefficients a_n and b_n for $n = 0, 1, \ldots, N/2$, using a fast Fourier transform (FFT) routine. The first of Eqs. (3) is performed to map the contour C on a z plane onto the contour $C^{(0)}$ on a w plane (Figs. 2a and 2b). The difference of the contour $C^{(0)}$ from the circle Γ now is $\mathcal{O}(\varepsilon^2)$. We change the contour $C^{(0)}$ to the unit circle Γ, resetting an error to zero. This operation is shown in Figs. 2b and 2c.

The second of Eqs. (3) is used to perform the exact inverse mapping $z = Z(w)$ of the unit circle Γ onto the contour $C^{(1)}$, which in

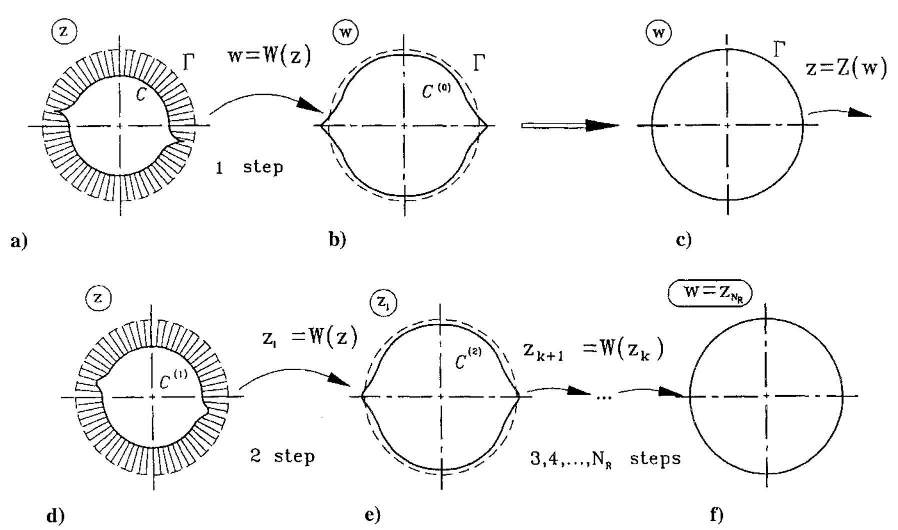


Fig. 2 R-procedure of the RT algorithm.

accordance to Lavrentiev principle differs from the initial contour C by small quantities of $\mathcal{O}(\varepsilon^2)$ (Figs. 2c and 2d).

Now, the first of the expressions (3) is used once more to perform the mapping of the contour $C^{(1)}$ (instead of the contour C) onto the contour $C^{(2)}$ (Figs. 2d and 2e). This procedure is repeated N_R times (Figs. 2e and 2f). The operation is completed when the rms error Δ_R falls below a prescribed tolerance ε_R:

$$\Delta_R \leq \varepsilon_R, \qquad \Delta_R^2 = \frac{1}{2\pi} \int_0^{2\pi} [\delta(\varphi)]^2 \, \mathrm{d}\varphi \tag{5}$$

This process, called the R-procedure, is described as $z = \Phi(w)$, $w = F[z, C^{(1)}]$. The sequence of steps $2, \ldots, N$ is identical in a certain sense to the classic Theodorsen mapping. However, whereas in the Theodorsen mapping the error in the polar angle is minimized, in the RT algorithm the error in the radius is minimized.

This peculiarity of the RT algorithm is of utmost importance. It allows complete solution of the problem of local singularities of domains with piecewise smooth contours and an arbitrary number of corner points. Corner smoothing is provided automatically since the corresponding process is implemented in the R-procedure and, as a consequence, in the T-procedure.

The preliminary domain transformations, such as Joukowski or Kármán–Trefftz (see Sec. IV), for the RT algorithm are useful but not absolutely necessary because they are applied for another purpose, i.e., to improve the domain structure as a whole for the following numerical mapping, not just to eliminate corner singularities. In fact, the R-procedure allows us to obtain an exact inverse transform and a quasi-exact direct transform of the contour $C^{(1)}$ that is very close to the given contour C, but the contour $C^{(1)}$ is smoother than the initial contour C because a finite Fourier series is used to obtain $C^{(1)}$. In practice, an error less than $\varepsilon_R = 10^{-6}$ can be normally reached in $N_R = 4$–5 steps on a double-precision computer.

IV. Recursive Mapping of an Arbitrary Domain (T-Procedure)

The T-procedure is a recursive step-by-step process of mapping an arbitrary domain onto a standard domain (a circle in the case of a simply connected domain and a circular annulus in the case of a doubly connected domain). The main idea of the T-procedure is the use of an auxiliary contour at each step of the recursive process. The choice strategy of the auxiliary contour is strictly dependent on the geometry and topology of the given contour and will be described later in detail.

Let us consider an arbitrary simply connected domain D of the z plane, bounded by piecewise smooth contour L. We are to build a conformal mapping of the domain D onto the unit circle D_w on the w plane with contour Γ_w in such a way that the specified point O of D lies over the center of this circle to effect the corresponding inverse mapping (Fig. 3a). Let us assume that the contour L is star-shaped in relation to the point O. This assumption is not fundamental and is used only for simplification of the algorithm description.

Let Γ_0 be a circle of radius ρ_0 with center at the origin O and the given contour L be embedded in the contour Γ_0. The relation $\rho_0 = 1$ could be reached using normalization of the variable z. A new auxiliary contour C_0, called a variation of the contour Γ_0, is constructed using the following equation in polar coordinates: $r(\varphi) = 1 - \delta_0(\varphi)$, where

$$\delta_0(\varphi) = \varepsilon[1 - r_0(\varphi)], \qquad \varepsilon = \frac{\varepsilon_0 \gamma_0}{\varepsilon_0 + \gamma_0}, \qquad \gamma_0 = \max[1 - r_0(\varphi)] \tag{6}$$

where $r_0(\varphi)$ is the polar equation of the contour L and ε_0 is any small number, usually about 0.05–0.2.

On the one hand, the auxiliary contour C_0 is close to the circle Γ_0 (because ε is a small number) and is an image of the given contour L. (In practice, a more sophisticated algorithm is used for construction of contour C_0, even though the two main properties of contour C_0 are the same.) As described before, the R-procedure for contour C_0 can now be used (Fig. 3b). As a result, two conformal mappings are obtained. The first one is the exact inverse mapping of the unit circle onto the contour $C_0^{(1)}$, which is very close to the contour C_0: $z = \Phi_0(z_1)$. The second one is a mapping of the contour $C_0^{(1)}$ onto the unit circle, $z_1 = F_0[z, C_0^{(1)}]$, accurate to within ε_R. As

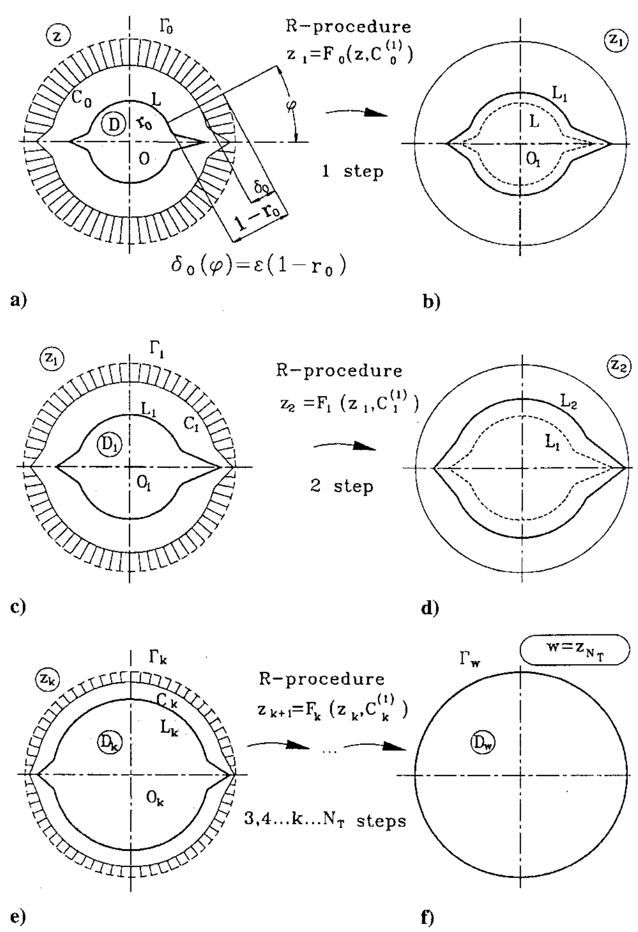


Fig. 3 T-procedure of the RT algorithm for a simply connected domain.

a result of the latter mapping, the given contour L is transformed onto the new contour L_1 (Fig. 3c). The contour L_1 is closer to the unit circle than the given contour L. This tendency is the basis for the successful recursive process construction, using the R-procedure at each step. Note that the replacement of the contour C_0 by the contour $C_0^{(1)}$ is not very important because both contours are close to one another and the contour C_0 was chosen arbitrarily to a certain degree.

At the second step of the T-procedure the contour L_1 is embedded into a new circle Γ_1 with a center at the point O_1, that is, the image of the center O. After the variable z_1 normalization, the circle Γ_1 has a unit radius, and the new auxiliary contour C_1 is built similarly to the contour C_0 (Fig. 3c). The contour C_1 is the variation of the circle Γ_1. Repeated use of the R-procedure gives the exact inverse mapping and approximately exact direct mapping of the contour $C_1^{(1)}$ onto the unit circle: $z_1 = \Phi_1(z_2)$; $z_2 = F_1[z_1, C_1^{(1)}]$.

The contour L_1 is transformed onto the contour L_2 (Fig. 3d). Repetition of the preceding operations gives the recursive process, called T-procedure (Figs. 3e and 3f):

$$\begin{aligned} z_k &= \Phi_k(z_{k+1}), & z_{k+1} &= F_k\big[z_k, C_k^{(1)}\big] \\ k &= 0, 1, \ldots, N_T, & z_0 &\equiv z, \qquad z_{N_T} = w \end{aligned} \tag{7}$$

The recursive process is completed when the rms distance Δ_{T_k} between the deformed contour L and the unit circle falls below a prescribed tolerance ε_T:

$$\Delta_{T_k} \leq \varepsilon_T, \qquad \Delta_{T_k}^2 = \frac{1}{2\pi} \int_0^{2\pi} [1 - r_k(\varphi)]^2 \, \mathrm{d}\varphi \tag{8}$$

Usually an error less than $\varepsilon_T = 10^{-4}$–10^{-5} can be reached in $N_T = 10$–50 steps on a double-precision computer.

Let us consider a doubly connected domain D of the z plane, bounded by two closed nonintersecting piecewise smooth lines L_{10} and L_{20} (Fig. 4a). The path-tracing direction of the inner contour L (clockwise) must be opposite to the path-tracing direction of the

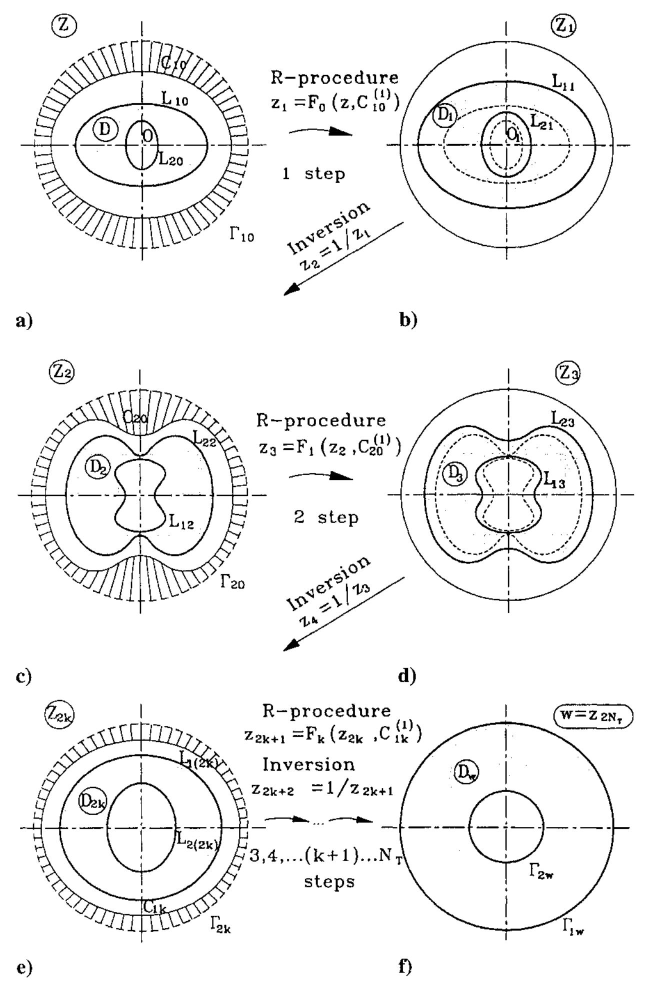


Fig. 4 T-procedure of the RT algorithm for a doubly connected domain.

outer contour L (anticlockwise as positive). The aim is conformal mapping of the given domain D onto a circular annulus D_w of the w plane in such a way that the line L_{10} transforms into the unit circle Γ_{1w}, the line L_{20} transforms into a circle Γ_{2w} of a smaller radius, and a specified point O of the z plane lies over the center of these circles (Fig. 4f) and in the inverse mapping as well. As before, the corresponding mapping is defined as an arbitrary angle of rotation of D relative to the point O.

The first step is the same as the first step of the T-procedure for a simply connected domain, bounded by the contour L_{10}. Namely, the contour Γ_{10} is the boundary of the unit circle, containing the normalized given domain D with the outer contour L_{10}. The new auxiliary contour C_{10} is built using Eqs. (6), where $r_0(\varphi)$ is the polar equation of the contour L_{10}. Using the R-procedure as described earlier, the transformation onto the z plane may be constructed: $z = \Phi_0(z_1)$, $z_1 = F_0[z, C_{10}^{(1)}]$. The contour $C_{10}^{(1)}$, which is very close to the contour C_{10}, goes over the unit circle. The contours L_{10} and L_{20} are transformed onto the contours L_{11} and L_{21}, respectively (Fig. 4b).

The next stage is the inversion with respect to the unit circle, $z_2 = 1/z_1$. The image L_{12} of the outer contour L_{11} is the inner contour; the image L_{22} of the inner contour is now the outer contour (Fig. 4c). After inversion, the path-tracing direction of the new outer contour L_{22} is positive (anticlockwise) as before, because the initial orientation of the contour L_{20} was negative. Once more the R-procedure may be used for the contour L_{22} (Fig. 4d): $z_2 = \Phi_1(z_3)$; $z_3 = F_1[z_2, C_{20}^{(1)}]$. The second step of the T-procedure for the doubly connected domain is defined in this way. At this step the auxiliary contour C_{20} is the variation of the unit circle Γ_{20} with respect to the contour L_{22}. The contour $C_{20}^{(1)}$, which is very close to C_{20}, is the unit circle image obtained by an inverse mapping Φ_1. An additional inversion with respect to the unit circle, $z_4 = 1/z_3$, completes the two first steps of the T-procedure. After it, the image L_{14} of the initial outer contour L_{10} becomes the outer contour, and the image L_{24} of the initial inner contour L_{20} becomes the inner contour.

The T-procedure for the doubly connected domain is a recursive process. Two typical steps of this are

$$(k+1)\text{-step: } z_{2k} = \Phi_k(z_{2k+1}) \qquad z_{2k+1} = F_k\left[z_{2k},\ C_{1k}^{(1)}\right];$$
$$z_{2k+2} = \frac{1}{z_{2k+1}}$$
$$(k+2)\text{-step: } z_{2k+2} = \Phi_{k+1}(z_{2k+3}) \tag{9}$$
$$z_{2k+3} = F_{k+1}\left[z_{2k+2},\ C_{2k}^{(1)}\right]; \qquad z_{2k+4} = \frac{1}{z_{2k+3}}$$

where $k = 0, 2, \ldots, N_T$; $z_0 \equiv z$; $z_{2N_T} \equiv w$. At each odd step the problem of the image $L_{1(2k)}$ of the outer contour L_{10} is solved; similarly, at each even step the problem of the image $L_{2(2k+2)}$ of the inner contour L_{24} is constructed. The T-procedure is completed when the rms differences of both contours (outer and inner) are smaller than the prescribed tolerance ε_T (Figs. 4e and 4f).

The convergence of the T-procedure for the doubly connected domain strictly depends on the relative width H of the initial noncircular annulus D:

$$H = \frac{1}{2\pi}\int_0^{2\pi}\left[1 - \frac{r_{20}(\varphi)}{r_{10}(\varphi)}\right]^2 \mathrm{d}\varphi \tag{10}$$

where r_{10} and r_{20} are the radius vectors of points of the lines L_{10} and L_{20}, respectively. If these contours are too close ($H \ll 1$), each step may destroy the result of the previous step. Practically, for the stable convergence of this process the relation $H \geq 0.2$ has to be satisfied.

The convergence of the T-procedure for domains with smooth boundary curves is very fast. For example, the domains considered by Wegmann[5,6] (inverted ellipse and eccentric circular annulus) can be mapped by the RT algorithm with $\varepsilon_T = 10^{-5}$–10^{-6} in 5–10 steps only. Besides the RT algorithm has a few additional advantages: 1) the possibility to map complicated domains with an arbitrary number of boundary corner points (up to a few hundreds); 2) the possibility to construct direct and inverse transforms simultaneously (this is important for generation of orthogonal grids as well as for solving of boundary value problems); and 3) the absence of matrix operations but multiple use of the FFT procedure.

As a rule, in hydrodynamics problems researchers are interested in inverse mapping $z = \Phi(w)$. This mapping could be presented as a superposition of the mappings $z_{k-1} = \Phi_k(z_k)$ and $k = N_T, \ldots, 1$. If each of the mappings is represented by a Taylor series with 64 terms, then for full implementation of the function $z = \Phi(w)$ only $129N_T$ real terms ($a_0^k, a_n^k, b_n^k, n = 1, \ldots, 64; k = N_T, \ldots, 1$) must be retained. Because of the moderate number of coefficients needed to describe the full procedure and the wide application of the FFT algorithm, we can use ordinary personal computer, such as an IBM personal computer. At the same time the high power of the series in the RT algorithm makes possible the mapping of very complex contours, including inner corners, ribs, and tracks.

V. Preliminary Analytical Transforms

One important advantage of the RT algorithm is its universality. In principle, it may be used for arbitrary domains with general shapes having piecewise smooth bounds. However, certain preliminary analytical transforms are useful to improve the efficiency of the method and to reduce the number of steps of the iterative procedures. The main idea of these transforms is to modify a given domain to a shape suitable for the RT algorithm. The rigorous definition of the requirements for such a shape is not easy, but the modified area should look like a noncircular annulus or deformed circle. For a simply connected domain it means that the ratio of the radii of the tangent circles interior and exterior to the given contour L, $\min r_0(\varphi)/\max r_0(\varphi)$, has to be about 0.3–1.0, where $r = r_0(\varphi)$ is the polar equation of the contour L. For a doubly connected domain the ratio of the inner and outer circle radii of the circular annulus, containing the given domain, should be about 0.5–0.8.

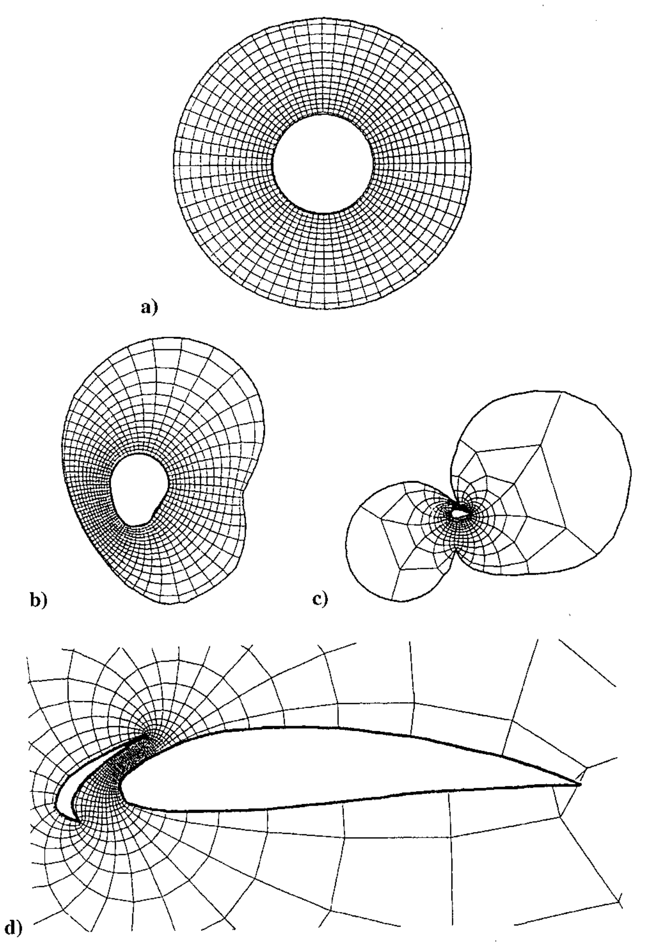


Fig. 5 Conformal mapping of a cross section of an airfoil with a slat (doubly connected domain) onto a circular annulus and a regular grid conformally equivalent to those of the polar coordinate system: a) the final domain after conformal mapping, b) the domain c after two Joukowski transforms, c) the domain d after the bilinear transform, and d) the initial domain.

The problem of an improvement of a given domain by using preliminary transforms is a nontrivial heuristical problem, which may be solved interactively. The various analytical transforms could be used for this purpose, in particular, the following: shift and rotation, inversion, bilinear transform, square root transform, and Joukowski or Kármán–Trefftz transforms with singularities located outside or inside of the given area. The later two transforms are widely used to remove corner singularities, in particular, in different modifications of the Theodorsen mapping.[7,12] In the case of the RT algorithm application they significantly reduce the number of necessary T-procedure steps.

An example demonstrating the mapping of a doubly connected external domain onto a circular annulus is presented in Fig. 5. The initial domain, the intermediate domains (after bilinear and Joukowski transforms), and the final domain after the completion of the mapping procedure are shown. The figure presents the regular grid conformally equivalent to those of the polar coordinate system. It is obtained by means of the inverse transform.

The preliminary transform changing the domain topology was found to be quite effective for elongated simply connected domains. The problem of a fluid stream in a rotating cylindrical tank with a parallelogram cross section may be considered as an obvious example. The Joukowski mapping is used to transform two copies of the initial domain onto a curvilinear annulus. These two copies have a common artificial cut and lie on two Riemann surface sheets. The singularities in this case are placed into the domain (Fig. 6). The RT algorithm is used to map the curvilinear annulus onto the circular annulus. The boundary conditions at the two contours are the same as those at the border of the given simply connected domain. The accuracy of this boundary value problem solution with the transform

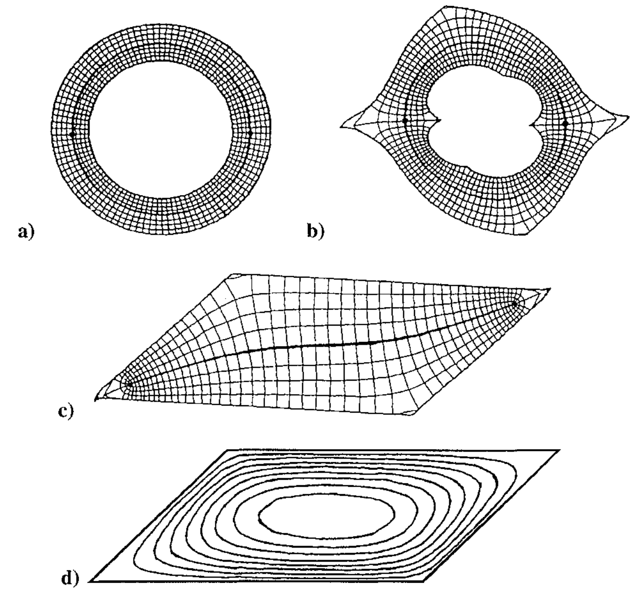


Fig. 6 Conformal mapping of a cross section of a parallelogram rotating tank (simply connected domain) onto a doubly connected domain with a preliminary topology change of the initial domain: a) the final circular annulus with an orthogonal grid, b) the domain c after the Joukowski transform with an orthogonal grid, c) the initial domain with an artificial cut and constructed orthogonal grid, and d) the flow lines in the initial domain.

onto an annulus (0.2%) was considerably higher than the solution constructed with the same number of iterative steps by the transform onto the unit circle without preliminary topology change (2%).

VI. Two-Dimensional Problems of Fluid Streams with Local Eddies

The problem of a flow around a two-dimensional analog of a spacecraft moving in a planetary atmosphere may be considered as a simple example demonstrating usefulness of the proposed method. The velocity field for this object can be constructed easily by the conformal mapping of the initial z plane domain (Fig. 7a) onto the w plane of an imagined stream around the unit circle (Fig. 7b).

The preliminary Joukowski transform was used to open the sharp angles, then the RT procedure was implemented to construct the final image. The location of the eddies and corresponding circulation at the w plane were determined by the Föppl steady-state condition[22] and Kutta–Joukowski condition at the sharp brake edges.[23] The inverse transform of the w plane (Fig. 7b) onto the z plane (Fig. 7a) gave an answer to the problem.

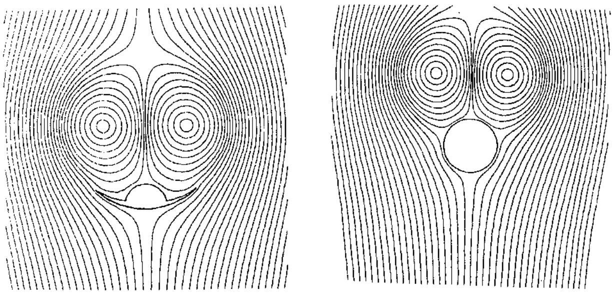
A more complicated example is a problem of internal circulating fuel flow with local eddies in a tank of a spacecraft. The liquid fuel is considered to be ideal liquid inside a circular cylinder with an even number K of equidistant damping ribs having the relative width $B = b/r_0 < 1$ (where r_0 is the inner cylinder radius). Supposing $r_0 = 1$, the region S of the fluid column cross section in the z plane was mapped onto the unit circle S^* in the w plane, so that the image of the point $z = 0$ was $w = 0$. An orthogonal grid on the z plane, obtained by the inverse mapping from an isothermal polar grid on the w plane, is presented in Fig. 8.

Let us consider two models.

1) The velocity field is formed by the central eddy and free rib eddies, each in the region of the sharp rib edge. The eddies' coordinates for this model are determined by the steady-state and Kutta–Joukowski conditions at the sharp rib edges.

2) The velocity field is caused directly by the rotation of a cylinder around a longitudinal axis with an angular velocity Ω ($\Omega > 0$ for anticlockwise rotation).

The circulation relayed to the rib eddies for both models was found by using the Kelvin circulation theorem.[23] The computed nondimensional attached moments of inertia for the first (I_Γ) and second (I_Ω) models for various values K and B are presented in the Table 1. The same computations were also made for similar models but without these rib eddies. Figures 9 and 10 show the streamlines



a) Actual flow in z plane b) w image of the flow

Fig. 7 Flow around a two-dimensional analog of a spacecraft with an aerodynamic brake moving in a planetary atmosphere.

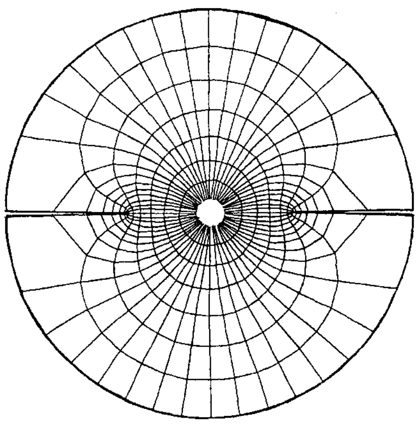
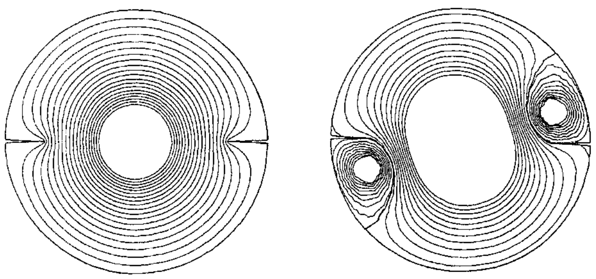
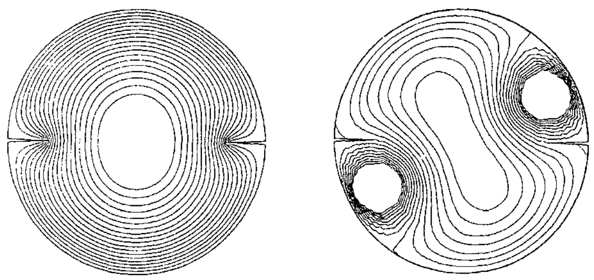


Fig. 8 Orthogonal grid obtained by inverse conformal mapping for a cross section of a circular tank with ribs.



a) Without vorticity regions b) With additional rib eddies

Fig. 9 Flow lines in a tank with inner ribs and a central eddy obtained using the RT algorithm of conformal mapping.



a) Without vorticity regions b) With additional eddies

Fig. 10 Flow lines in a rotating tank with ribs obtained using the RT algorithm of conformal mapping.

of the flow in the z plane (the central eddy in Fig. 9 and the rotating cylinder in Fig. 10) without vorticity regions (Figs. 9a and 10a) and with them (Figs. 9b and 10b). The conformal mapping was performed by means of the RT algorithm with preliminary Joukowski transforms. The computed value of I_Ω for $K = 4$ and $B = 0.3$ was negative; that must be interpreted as an inadequacy of the model used in that case.

Table 1 Attached nondimensional moments of inertia of fluid in a circular cylinder with inner ribs

Number of ribs, K	Relative width, B	Central eddy, I_Γ	Rotating cylinder, I_Ω	Without eddies: Analytical I	Without eddies: Numerical I_{RT}
4	0.3	0.012	——	0.412	0.417
4	0.4	0.304	0.189	0.657	0.645
6	0.3	0.331	0.288	0.586	0.572
6	0.4	0.645	0.622	0.864	0.845

Table 2 Typical accuracy characteristics of numerical computations using the RT algorithm of conformal mapping

Parameter	Smooth contour	Contour with corners
Given tolerance for the R-procedure, ε_R	10^{-5}–10^{-6}	10^{-5}–10^{-6}
Number of steps of the R-procedure, N_R	3–5	3–5
Given tolerance for the T-procedure, ε_T	10^{-4}–10^{-5}	10^{-3}–10^{-4}
Number of steps of the T-procedure, N_T	10–30	30–50
Relative computational error for integral characteristics (moment of inertia, etc.), Δ_I	10^{-3}–10^{-6}	10^{-2}–10^{-3}
Relative computational error for velocity field characteristics, Δ_V	10^{-2}	——

The moment of inertia of the fuel in the rotating cylinder without eddies may be estimated analytically. The corresponding exact values (I), obtained by Dokuchayev,[24] are presented in Table 1 together with the similar values (I_{RT}) computed numerically using the RT algorithm of conformal mapping. The agreement is quite satisfactory; the mean relative error of numerical computations is only about 2%.

The RT algorithm was also found to be a very efficient means to compute eigenmodes in artificial and natural bodies. In particular, it was successfully used to compute nonrotational and rotational eigenoscillations (seiches) in the Caspian Sea.[25]

VII. Discussion and Conclusions

The efficiency and accuracy of the RT algorithm of conformal mapping for smooth and piecewise smooth contours are illustrated in the Table 2. Typical values of rms errors of the R- and T-procedures (ε_R and ε_T) are presented together with the corresponding numbers of the iterative steps (N_R and N_T). Normally, the values ε_R and ε_T are taken as initial parameters, and necessary numbers of the steps N_R and N_T to achieve these tolerances are estimated a posteriori.

Typical values of relative computational errors of the Stokes–Joukowski problem, Δ_I and Δ_V, are also presented in the Table 2. These values are estimated as $\Delta_I = \Delta I/I$ and $\Delta_V = \Delta V/V$, where I and V are the exact values of the attached moment of inertia and relative fluid velocity, and Δ_I and Δ_V the errors related to their numerical computation. It is clear that these errors are quite small.

All computations described earlier were made with double precision on the IBM personal computer AT 386/387 using a version of the software package RT-Soft elaborated by the authors. The maximum number of contour points in RT-Soft is up to 1024, and the maximum number of terms of the Taylor series in the R- and T-procedures is 64. The typical computational time of the conformal mapping and generation of orthogonal grid is from 3–5 to 30–40 min, depending on complexity of the initial domain.

Summarizing the results, we can conclude that the RT algorithm has proved to be highly suitable for conformal mapping of complicated simply and doubly connected domains using computers of modest memory and performance characteristics. In particular, it may be very appropriate for generation of orthogonal grids. At the same time, numerical methods based on the RT algorithm of conformal mapping may be competitive for problems of mathematical physics with the finite difference method (FDM) and the finite element method (FEM). There are even certain advantages in numerical methods based on conformal mappings, as in the problem of adequate discrete approximation of a complicated bound line. However, these methods and, particularly, the RT algorithm are rather a supplement than an alternative to FDM and FEM. The combination of the RT algorithm with FDM and FEM was found to be very effective (after conformal mapping of the initial complicated domain, the ordinary procedures of FDM and FEM are used to solve the corresponding problems for new simplified areas). Apparently, the RT algorithm could be effectively used to solve certain two-dimensional problems in the same way as the Runge–Kutta method is used to solve one-dimensional problems.

In the present paper the capabilities of the recursive numerical RT algorithm of conformal mapping were applied to solve certain two-dimensional hydrodynamics boundary value problems. Two models of flow in a circular tank with inner ribs and vorticity regions were simulated numerically. Good agreement was found between analytical and numerical results.

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